Geodesics in the Brownian map : Strong confluence and geometric structure

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Plan

1. Lecture 1
   - Background and motivation
   - Strong confluence of geodesics and intersection behavior
   - Construction of the Brownian map by the Brownian snake
2. Lecture 2
   - Breadth-first exploration of the Brownian map
   - Proof of strong confluence and intersection behavior
   - Finite number of geodesics
3. Lecture 3
   - Exponent for splitting points
   - Geodesic stars
   - Topology of geodesics between a pair of points
   - Approximation by geodesics between typical points
   - Open questions

The Brownian map

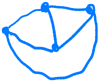
General idea

The Brownian map is the "canonical" model for a metric space chosen "uniformly at random" among metric spaces which have the topology of the two-dimensional sphere $\mathbb{S}^2$.

Gromov-Hausdorff scaling limit of a large class of planar maps chosen uniformly at random.

- Triangulations and $2p$-angulations with n faces [Le Gall '13]
- Quadrangulations with n faces [Miermont '13]
- Bipartite planar maps, random simple triangulations and quadrangulations, ... [Abraham, Addario-Berry, Albenque, Bettinelli, Jacob, Miermont, ...]

Denoted by $(\mathcal{S}, d, \nu)$.



Approximation by quadrangulation

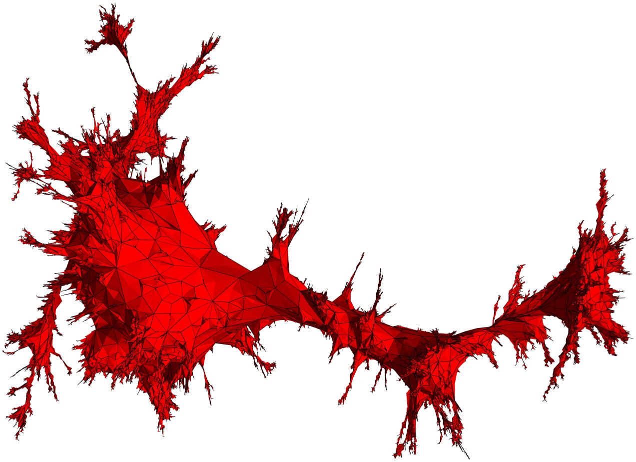


Image by Jérémie Bettinelli

The Brownian map

- Homeomorphic to the sphere $\mathbb{S}^2$ [Le Gall and Paulin '08] (also see a later proof [Miermont '08])
- Hausdorff dimension equal to 4 [Le Gall '07]

$$\mathcal{H}^d(S) = \lim_{r\to 0} \inf \Big\{ \sum_i r_i^d : \text{ countable cover of } S \text{ by balls with radii } r_i < r \Big\}$$

$$\dim_{\mathrm{H}}(S) = \inf \big\{ d \geq 0 : \mathcal{H}^d(S) = 0 \big\}.$$

- Equivalent as a metric measure space to $\sqrt{8/3}$-LQG (Liouville quantum gravity) [Miller and Sheffield '16 and '20]. The Brownian map can be canonically embed into $\mathbb{S}^2$.

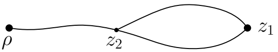
Classification of all geodesics to the root

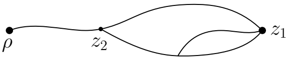
[Le Gall '10]

Let $\rho \in \mathcal{S}$ be a distinguished point called the root. **The law of $(\mathcal{S}, d, \nu, \rho)$ is invariant if we resample ρ independently according to ν.**

The following facts hold a.s.

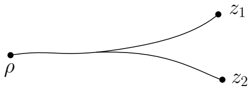
- For ν-a.e. point $z \in \mathcal{S}$, there is a unique geodesic between z and ρ.
- Every point in $\mathcal{S}$ is connected to ρ by at most 3 geodesics. The set of points connected by 2 or 3 geodesics to ρ is dense in $\mathcal{S}$ and has ν-mesure zero.
- The set of points connected by 2 geodesics to ρ has dimension 2. The set of points connected by 3 geodesics to ρ has dimension 0, and is countable.





Confluence of geodesics at the root

[Le Gall '10]



This plays a major role in the works that identify the Brownian map as the scaling limit of uniform random maps [Le Gall] and [Miermont], as well as in the proof of the equivalence of $\sqrt{8/3}$-LQG with the Brownian map [Miller and Sheffield].

The results for ρ applies for ν-a.e. point (also called **typical point**).

Geodesics between exceptional points ?

[Angel, Kolesnik and Miermont '17] (j, k)-normal network

FIGURE – A $(3, 2)$-normal network

- The set of pairs of points connected by a (j, k)-normal network is non-empty if and only if $j, k \in \{1, 2, 3\}$.
- The set of pairs of points connected by a (j, k)-normal network has dimension $12 - 2(j + k)$. $j=k=1$ ⟿ dim $= 8 = S \times S$
- The set of pairs of points connected by a $(3, 3)$-normal network is dense and countable.

These results do not rule out the existence of other exceptional points between which the collection of geodesics has a topology which is not that of a normal network. In fact, there do exist such other points.
AKM also proves a strong version of the confluence of geodesics : this version is also associated with typical points and does not apply to all geodesics.

All geodesics in the brownian map

The best results were obtained by Le Gall and AKM. No result at all about **all** geodesics at the same time. The most basic questions were unknown.

- Are there pairs of points which are connected by infinitely many geodesics ?
- Is there any point from which infinitely many disjoint (except at the starting point) geodesics emanate ?

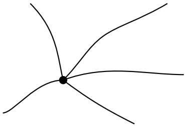


FIGURE – A geodesic star

- What topology of geodesics can there be between two points ?

Our goal is to answer these questions and to provide a global description of the behavior of **all** geodesics at the same time.

Different versions of the Brownian map

There are different versions of the Brownian map.

- Often, the term Brownian map refers to the standard unit area Brownian map $(\mathcal{S}, d, \nu)$ with $\nu(\mathcal{S}) = 1$. It has a probability distribution which we denote by $\mu_{\mathrm{BM}}^{A=1}$.
- There is also an infinite measure μ_{BM} on $(\mathcal{S}, d, \nu)$, where $\nu(\mathcal{S}) \in (0, \infty)$ is not fixed, but has an infinite distribution. Conditioning μ_{BM} on $\nu(\mathcal{S}) = a$, we have $\mu_{\mathrm{BM}}^{A=a}$.
- **Invariance by scaling :** If $(\mathcal{S}, d, \nu)$ has law $\mu_{\mathrm{BM}}^{A=1}$, then if we multiply its distance by $a^{1/4}$ and its area par a, we obtain $\mu_{\mathrm{BM}}^{A=a}$.

We expect that our results can also be transferred to other Brownian surfaces.

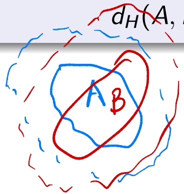
Strong confluence of geodesics

- The confluence of geodesics at the root does not occur for all points of the Brownian map.
- We show that a different form of the confluence of geodesics phenomenon which holds simultaneously for all geodesics in the Brownian map.

Definition (Hausdorff distance)

Let X be a metric space. For all $A \subseteq X$ and $\varepsilon > 0$, let $A(\varepsilon) = \cup_{x \in A} B(x, \varepsilon)$ be the ε-neighborhood of A. The Hausdorff distance between two closed sets $A, B \subseteq X$ is defined to be

$$d_H(A, B) = \inf\{\varepsilon > 0 : A \subseteq B(\varepsilon),\ B \subseteq A(\varepsilon)\}.$$



Strong confluence of geodesics

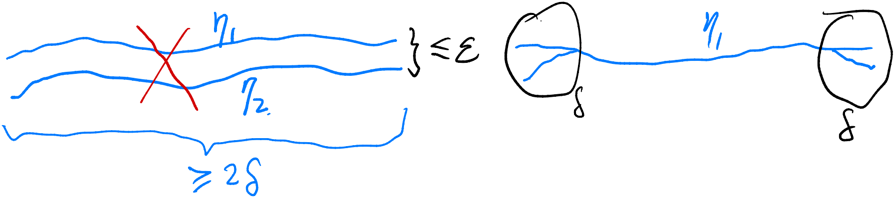
Theorem 1 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. For each $u > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the following holds. Let $\delta = \varepsilon^{1-u}$. Suppose that $\eta_i : [0, T_i] \to \mathcal{S}$ for $i = 1, 2$ are two geodesics with $T_i = d(\eta_i(0), \eta_i(T_i)) \geq 2\delta$ and

$$d_H(\eta_1([0, T_1]), \eta_2([0, T_2])) \leq \varepsilon,$$

then

$$\eta_i([\delta, T_i - \delta]) \subseteq \eta_{3-i} \text{ for } i = 1, 2.$$



Strong confluence of geodesics (more precise version)

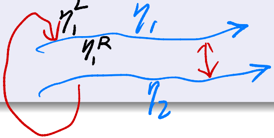
Definition (interior-internal metric)

Let (X, d) be a metric space and $S \subseteq X$. Let d_S be the interior-internal metric on S, whereby $d_S(u, v)$ is given by the infimum of the d-length of paths which are contained in the interior of S, except possibly their endpoints.

Definition (One-sided Hausdorff distance)

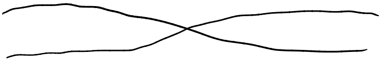
Let η_1, η_2 be two geodesics of $(\mathcal{S}, d, \nu)$. Then $\mathcal{S} \setminus \eta_1$ is a simply connected set whose boundary is the union of the left and right sides of η_1, which we denote by η_1^L and η_1^R. Let ℓ_L (resp. ℓ_R) be the Hausdorff distance between η_1^L (resp. η_1^R) and $\eta_2 \setminus \eta_1$ with respect to the interior-internal metric $d_{\mathcal{S} \setminus \eta_1}$. We define the one-sided Hausdorff distance from η_1 to η_2 by

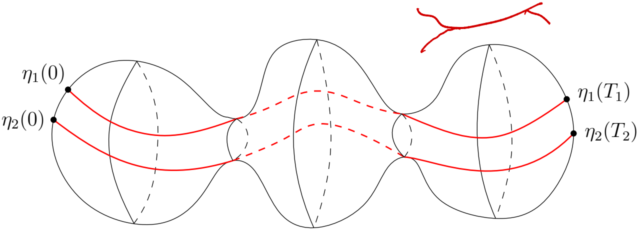
$$d_H^1(\eta_1, \eta_2) = \min(\ell_L, \ell_R).$$



We always have

$$d_H(\eta_1, \eta_2) \leq d_H^1(\eta_1, \eta_2).$$





$\eta_1(0)$
$\eta_2(0)$
$\eta_1(T_1)$
$\eta_2(T_2)$

Strong confluence of geodesics (more precise version)

Theorem 2 (Miller, Q. '20)

There exists $c > 0$ such that the following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. There exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the following holds. Let $\delta = c\varepsilon \log \varepsilon^{-1}$. Suppose that $\eta_i : [0, T_i] \to \mathcal{S}$ for $i = 1, 2$ are two geodesics with $T_i = d(\eta_i(0), \eta_i(T_i)) \geq 2\delta$ and

$$d_H^1(\eta_1([0, T_1]), \eta_2([0, T_2])) \leq \varepsilon,$$

then

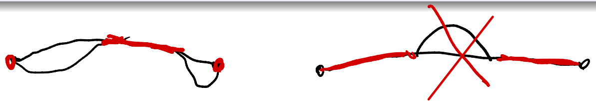
$$\eta_i([\delta, T_i - \delta]) \subseteq \eta_{3-i} \text{ for } i = 1, 2.$$

We believe that the order of magnitude $\varepsilon \log \varepsilon^{-1}$ is optimal in Theorem 2.

Intersection behavior of geodesics

Theorem 3 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. Suppose that $\eta_i : [0, T_i] \to \mathcal{S}$ for $i = 1, 2$ are two geodesics, then $\{t \in (0, T_i) : \eta_i(t) \in \eta_{3-i}\}$ is connected for $i = 1, 2$.



- It is enough to consider the case where η_1 and η_2 do not cross each other.
- Theorem 2 + Theorem 3 + The fact that there are at most ε^{-u} bottlenecks along a geodesic $\implies$ Theorem 1.

Main idea

The proofs of previous results (e.g. [Le Gall '10] and [Angel, Kolesnik and Miermont '17]) primarily make use of the Brownian snake encoding of the Brownian map, see [Chassaing and Schaeffer '04], [Marckert and Mokkadem '06] and [Le Gall '07].

- Analogous to the Cori-Vauquelin-Schaeffer bijection for the quadrangulations.
- The Brownian map is constructed from a labeled continuous random tree (CRT). [Aldous '91, '93]

This corresponds to the **depth-first** exploration of the Brownian map. **This leads to very precise description of the geodesics to the root.**

We will discuss in the next lecture

Our work primarily make use of the **breadth-first** exploration of the Brownian map.

Brownian snake encoding of the Brownian map

Construction of the unit area Brownian map by the Brownian snake (X, Y).

- Let X be a Brownian excursion on $[0, 1]$.
- Let $\mathcal{T}$ be the continuum random tree (CRT) encoded by X.

$$d_X(s, t) = X_s + X_t - 2 \inf_{r\in[s,t]} X_r.$$

- Let Y be a mean-zero Gaussian process on $[0, 1]$ with covariance function

$$\mathrm{cov}(Y_s, Y_t) = \inf_{r\in[s,t]} X_r.$$

- For $s, t \in [0, 1]$, let

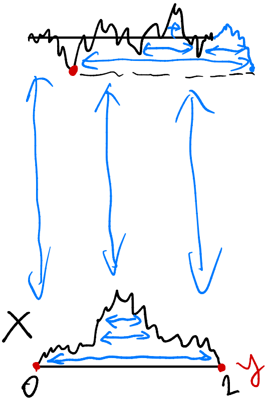
$$d^\circ(s, t) = Y_s + Y_t - 2 \max\left(\inf_{s\in[s,t]} Y_r, \inf_{s\in[t,s]} Y_r \right)$$

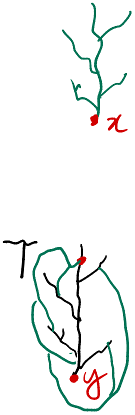
- The distance d° induces a distance $d^\circ_{\mathcal{T}}$ on $\mathcal{T}$. For $a, b \in \mathcal{T}$, let

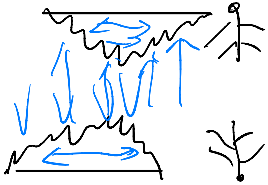
$$d(a, b) = \inf \left\{ \sum_{j=1}^{k} d^\circ_{\mathcal{T}}(a_{j-1}, a_j) \right\}$$

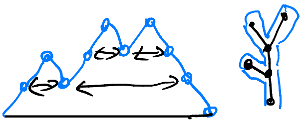
  where the infimum is over all $k \in \mathbb{N}$ and $a_0 = a, a_1, \cdots, a_{k-1} \in \mathcal{T}, a_k = b$.
- $a \sim b$ if $d(a, b) = 0$. Then $(\mathcal{S}, d, \nu) = \mathcal{T}/\sim$.

Brownian snake encoding of the Brownian map









Brownian snake encoding of the Brownian map

- The root x and the dual root y are distributed as two independently chosen points in $\mathcal{S}$ according to ν. Let $\mu_{\mathrm{BM}}^{A=1}$ denote the law of $(\mathcal{S}, d, \nu, x, y)$.
- The mesure μ_{BM} is constructed by
  - first choosing the time length of the Brownian excursion according to the infinite measure $ct^{-3/2}dt$
  - then sampling a Brownian excursion $(X_s)_{0\leq s\leq t}$ of time length t.
- The area of the map is given by the total length t of the excursion X.

1. Lecture 1
   - Background and motivation
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Breadth-first exploration of the Brownian map

Our work primarily make use of the **breadth-first** exploration of the Brownian map.

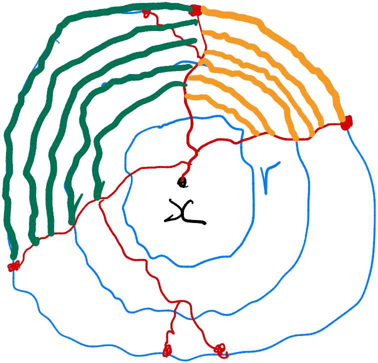
- Analogous to the peeling by layers of random planar maps. [Ambjørn, Durhuus, Jonsson and Jonsson '97], [Watabiki '95] and [Angel '03]
- Various aspects in the discret and in the continuum were developed by Bertoin, Budd, Curien, Kortchemski, Le Gall, Miller and Sheffield, and so on.

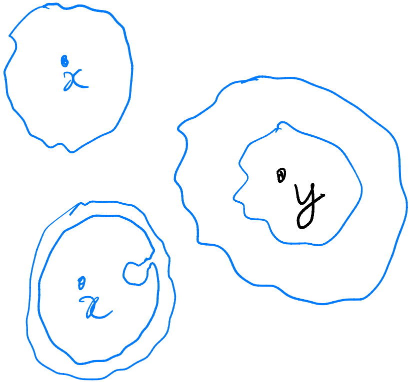
  We will in particular use the setting and results from [Miller and Sheffield '15] "An axiomatic characterization of the Brownian map"

Particularly amenable for establishing independence properties along geodesics.

Breadth-first exploration of the Brownian map

- Let $(\mathcal{S}, d, \nu, x, y)$ be sampled from μ_{BM}.
- Let $B^\bullet_y(x,r)$ be the metric ball of radius r centred at x and filled with respect to y. We can associate a boundary length L_r to $\partial B^\bullet_y(x,r)$.
- The process $(L_{d(x,y)-r}, 0 \le r \le d(x,y))$ is distributed as a continuous state branching process (CSBP) with parameter $3/2$.





Continuous state branching process (CSBP)

- Introduced in [Jiřina '58], also studied in [Lamperti '67]. Also see the more recent expository texts [Le Gall '99] and [Kyprianou '06].
- It is defined via the Lamperti transform. If (X_s) is an α-stable Lévy process with only upward jumps and

$$s(t) = \inf\left\{r > 0 : \int_0^r \frac{1}{X_u} du \geq t\right\},$$

then $Y_t := X_{s(t)}$ is an α-CSBP.

$x_1 \qquad x_2$

$X_1 + X_2$

- The transition kernel of Y satisfies

x_1+x_2

$$P_t(x_1 + x_2, \cdot) = P_t(x_1, \cdot) * P_t(x_2, \cdot).$$

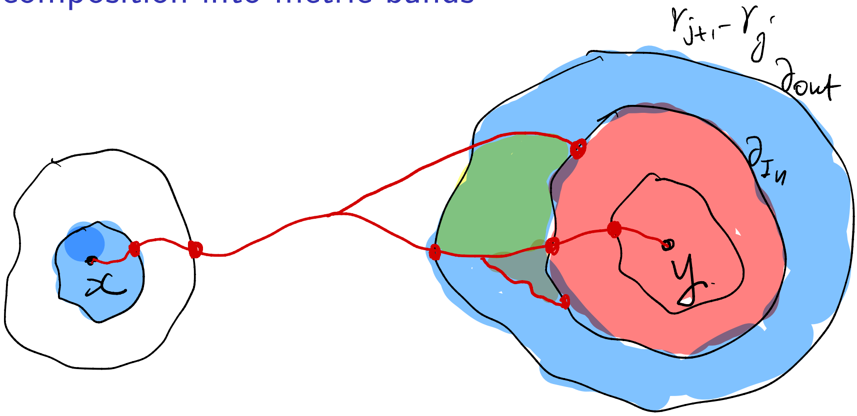
- $(Y_{C^{\alpha-1}t})$ is equal in distribution to (CY_t).
- One can also define an excursion measure for α-stable CSBP by doing the Lamperti transform to an α-stable Lévy excursion sampled as follows :
  - Pick a lifetime t from the infinite measure $t^{-1-1/\alpha} dt$
  - Given t, sample an α-stable Lévy excursion.

In the Brownian map $(\mathcal{S}, d, \nu, x, y)$ sampled from μ_{BM}, we have $t = d(x, y)$.

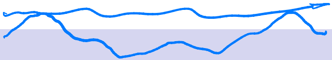
Decomposition into metric bands

- Fix $0 < r_1 < r_2 < \cdots < r_k$. $\mathcal{B}_j := B_y^\bullet(x, d(x,y) - r_j) \setminus B_y^\bullet(x, d(x,y) - r_{j+1})$ is a metric space with interior-internal metric $d_{\mathcal{B}_j}$ and the measure $\nu_{\mathcal{B}_j} := \nu|_{\mathcal{B}_j}$.
- On the event $d(x,y) > r_j$, $\mathcal{B}_j$ is non-empty, and is either an annulus if $d(x,y) > r_{j+1}$ or a topological disk if $d(x,y) \le r_{j+1}$.
- $\mathcal{B}_j$ is **independent** of $\mathcal{B}_1, \dots, \mathcal{B}_{j-1}$, conditionally on the length of $\partial_{\mathrm{In}} \mathcal{B}_j$.
- The boundary $\partial_{\mathrm{In}} \mathcal{B}_j$ is marked by the unique point visited by the unique geodesic between x and y. The quanity $r_{j+1} - r_j$ is called the width of $\mathcal{B}_j$.
- Let $\mathsf{P}_{\mathrm{Band}}^{L=\ell, W=w}$ be the probability measure on metric bands $(\mathcal{B}, d_{\mathcal{B}}, \nu_{\mathcal{B}}, z)$ with inner boundary length ℓ, width w and marked point z on the inner boundary.
- We can further cut out independent geodesic slices $(\mathcal{G}, d_{\mathcal{G}}, \nu_{\mathcal{G}}, z)$ with law $\mathsf{P}_{\mathrm{Slice}}^{L=\ell, W=w}$.

Decomposition into metric bands

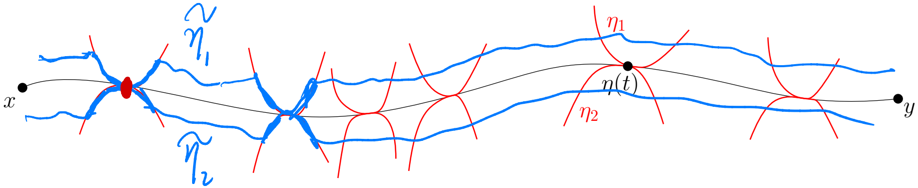


Close geodesics must intersect near their endpoints



A weaker version of the strong confluence

If two geodesics are sufficiently close with respect to the one-sided Hausdorff distance, then they should intersect each other near their endpoints.

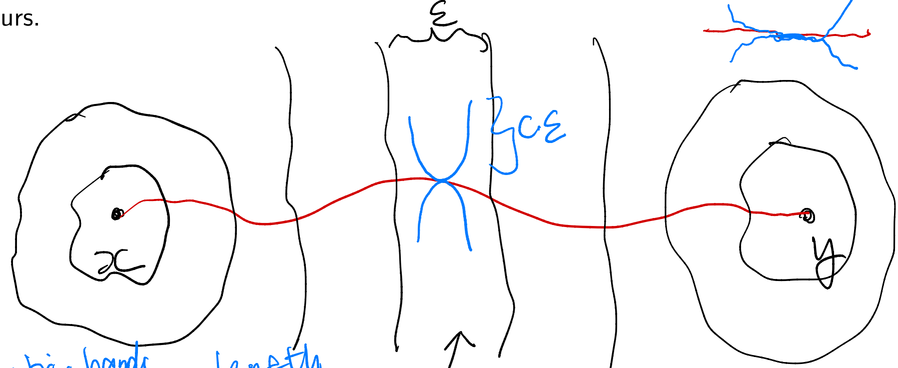


1. For two ν-typical points x, y, with overwhelming probability, there are many $\mathscr{X}$'s along the geodesic η between x and y. Every branch of an $\mathscr{X}$ is the **unique** geodesic between its endpoints.

2. If $\widetilde{\eta}$ crosses an $\mathscr{X}$ centred on $\eta(t)$, then $\widetilde{\eta}$ also intersects $\eta(t)$.
3. If $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$ are close to each other, then one can find a geodesic η between $\widetilde{\eta}_1$ and $\widetilde{\eta}_2$.

Close geodesics must intersect near their endpoints

Proof of point 1 : In each metric band, there is a positive probability that an x occurs.

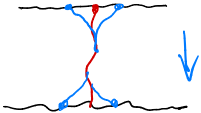


metric bands

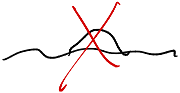
$C \log \varepsilon^{-1} \rightarrow C\varepsilon \log \varepsilon^{-1}$, length

$\mathbb{P} \geq p > 0$

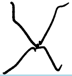
$(1-p)^{c \log \varepsilon^{-1}} = \varepsilon^{a} > \delta$



Proof roadmap of various results



Weak version of strong confluence



+ multiplicities
splitting points

Finite number of geodesics

Intersection behavior

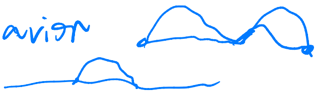
weak version of

Strong confluence

More precise results on the geometric structure of geodesics (next lecture)

Proof of Theorem 3

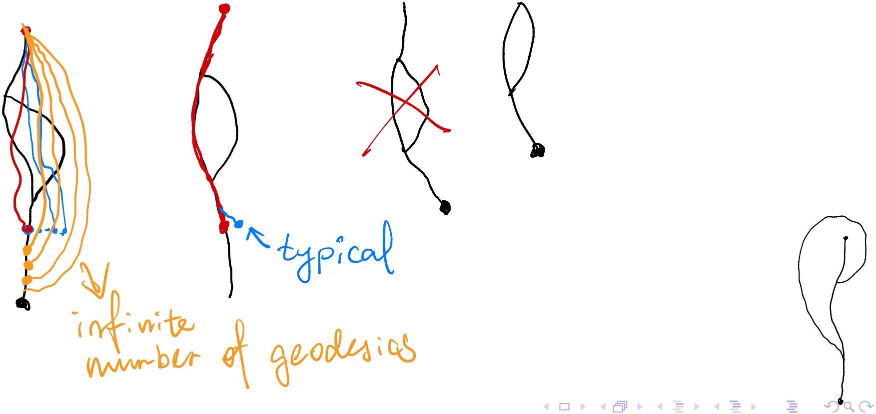
intersection behavior



Using the following input.

Finite number of geodesics (weak version of later results)

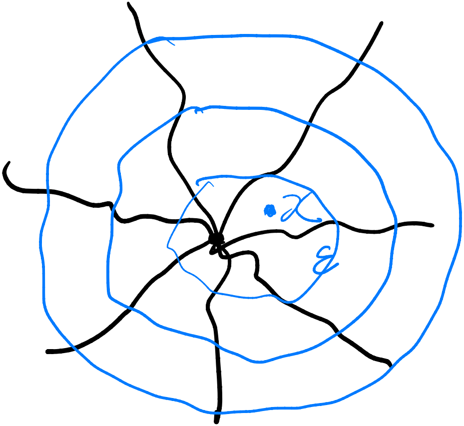
There exists $C < \infty$, such that the number of distinct geodesics between any pair of points is at most C.



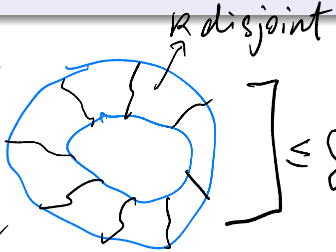
Finite number of disjoint geodesics from a point

Crude estimate (weak version of later results)

There exists $C < \infty$, such that the number of disjoint geodesics from any given point is at most C.



$\mathbb{P}\Big[$



$\Big] \le \delta$

$$\delta^{\log \varepsilon^{-1}} = \varepsilon^{a>4}$$

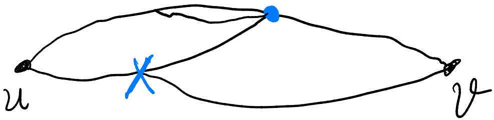
Finite number of geodesics between two points

Definition (Splitting point)

For $u, v \in \mathcal{S}$ distinct, we say that z is a splitting point from v to u of multiplicity at least k, if there exist $0 < r < t < d(u, v)$ and geodesics $\eta_1, \dots, \eta_{k+1}$ from v to u such that $\eta_i(t) = z$ for all $1 \leq i \leq k+1$ and

$$\eta_i([t-r, t]) = \eta_j([t-r, t]), \quad \eta_i((t, t+r]) \cap \eta_j((t, t+r]) = \emptyset$$

for all $1 \leq i < j \leq k+1$. The multiplicity of z is equal to the largest integer k such that the property above holds.



Finite number of geodesics between pairs of points

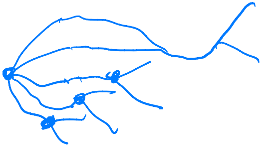
Proposition 4

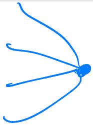
For μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$, the following holds for all $u, v \in \mathcal{S}$ distinct. The sum of the multiplicities of all the splitting times from v to u is at most 8.

Together with the rough estimate on the number of disjoint geodesics from a point, we can deduce the following.

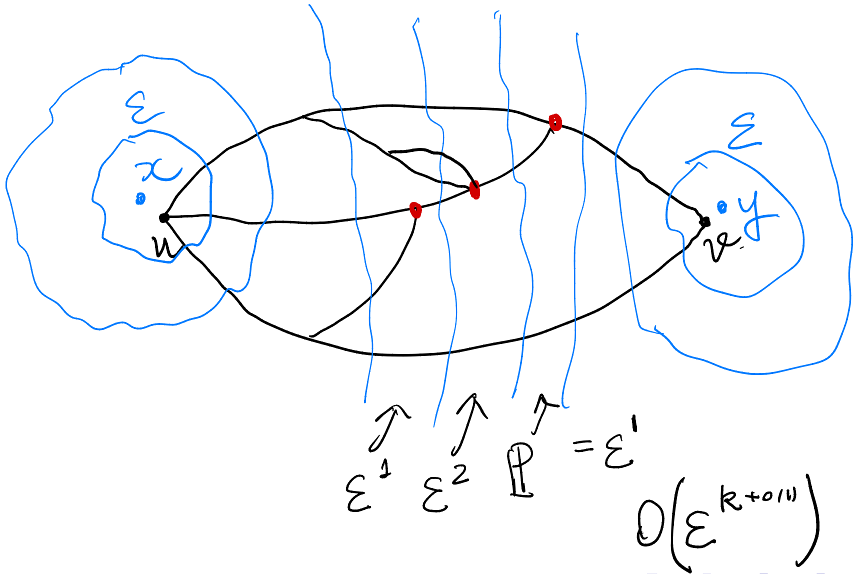
Finite number of geodesics between pairs of points

There exists a deterministic constant C so that the number of geodesics which connect u to v is at most C.

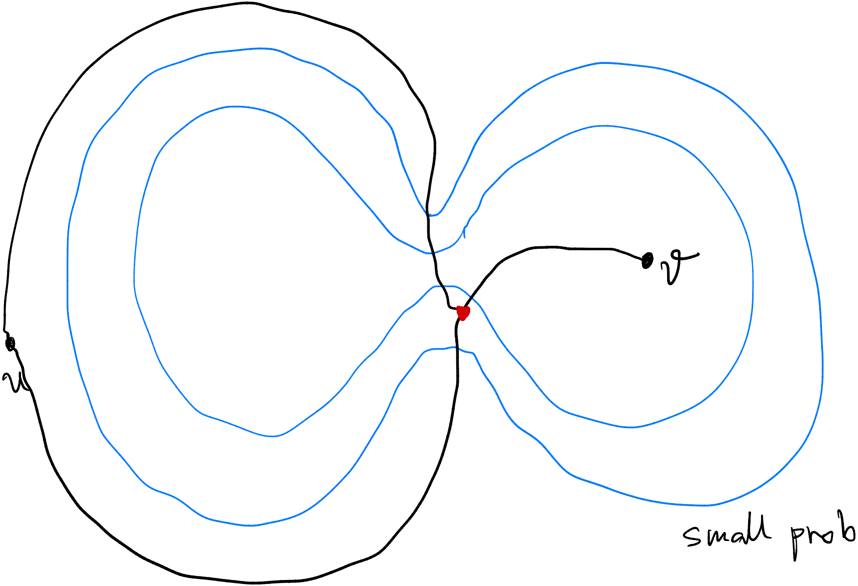




Exponent for splitting points



Exponent for splitting points



Geodesic stars

Let Ψ_k be the set of k-star points.

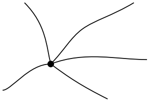


FIGURE – A 5-star point

Theorem 5 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. The set Ψ_k is empty for $k \geq 6$. For $1 \leq k \leq 5$, we have

$$\dim_{\mathrm{H}}(\Psi_k) \overset{=}{\leq} 5 - k.$$

- The k-star points played an essentiel role in the proof of the convergence of the quadrangulations towards the Brownian map in [Miermont '13]
- [Miermont '13] conjectured that there exist k-star points for $1 \leq k \leq 4$, and there do not exist k-star points for $k \geq 6$.
- The matching lower bounds were recently proved by Le Gall.
- It is still an open question whether there exist 5-star points.

Exponent for disjoint geodesics from a point

$\mathbb{P}$ [] $= O(\varepsilon^{k-1+o(1)})$

$X_1, X_2, \cdots, X_k$ CSBP

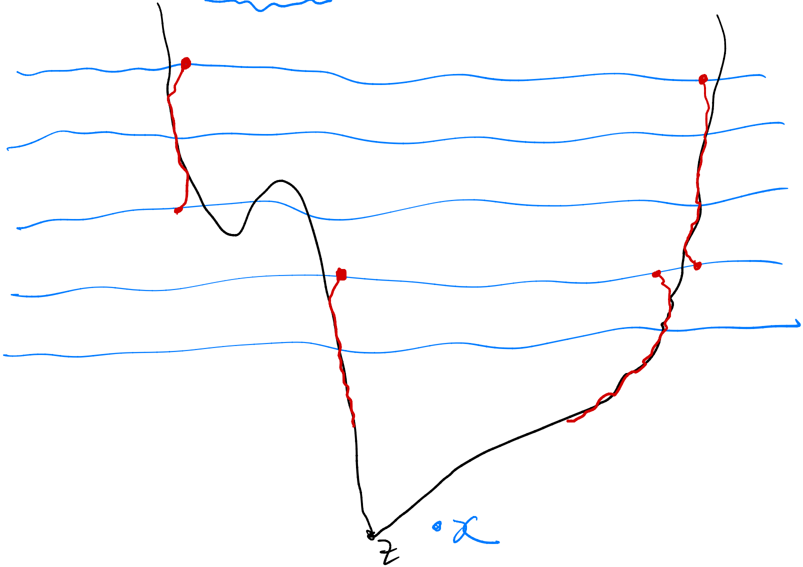
$T_1 \quad T_2 \quad \cdots, T_k$

$\mathbb{P}$

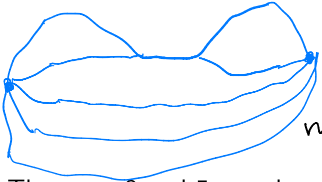
$= O(\varepsilon^{k-1+o(1)})$

Exponent for disjoint geodesics from a point

Need to take away some bad layers: non-merging layer, crossing layer, fat layer.



Topology of geodesics between a pair of points



not optimal.

- Theorems 3 and 5 together reduce the possible configurations of geodesics between any pair of points to a finite number of cases up to homeomorphism.
- We will further reduce the number of possible configurations, and then give a dimension upper bound for the set of pairs of points connected by each configuration (up to homeomorphism).

Topology of geodesics between a pair of points

Theorem 6 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. For all $u, v \in \mathcal{S}$ distinct, every geodesic from v to u contains at most two splitting points from v to u, and the multiplicity of each splitting point is 1. Let $\Phi_{I,J,K}$ be the set of (u, v) such that $u, v \in \mathcal{S}$ are distinct and there exists $r > 0$ so that the following holds.

1. *There are geodesics $\eta_1, \ldots, \eta_I$ from u to v such that the sets $\eta_i((0, r))$ for $1 \leq i \leq I$ are pairwise disjoint.*
2. *There are geodesics $\eta_1, \ldots, \eta_J$ from v to u such that the sets $\eta_i((0, r))$ for $1 \leq i \leq J$ are pairwise disjoint.*
3. *There are K splitting points from v to u.*

If $11 - (I + 2J + K) \geq 0$, then

$$\dim_{\mathrm{H}}(\Phi_{I,J,K}) \leq 11 - (I + 2J + K).$$

Otherwise $\Phi_{I,J,K} = \emptyset$.

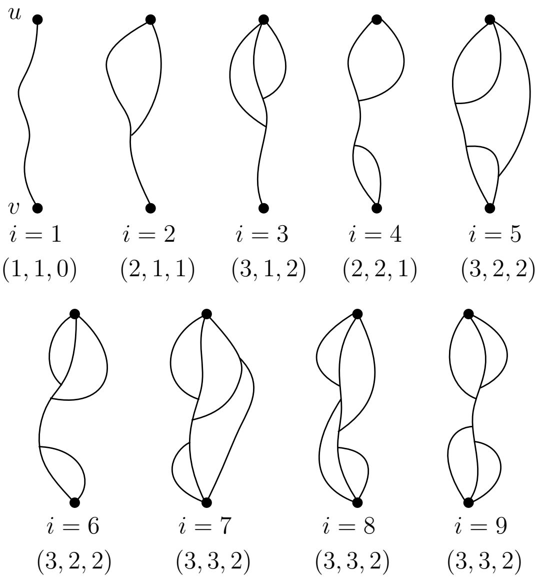


FIGURE – Optimal configurations and the associated triplets (I, J, K)

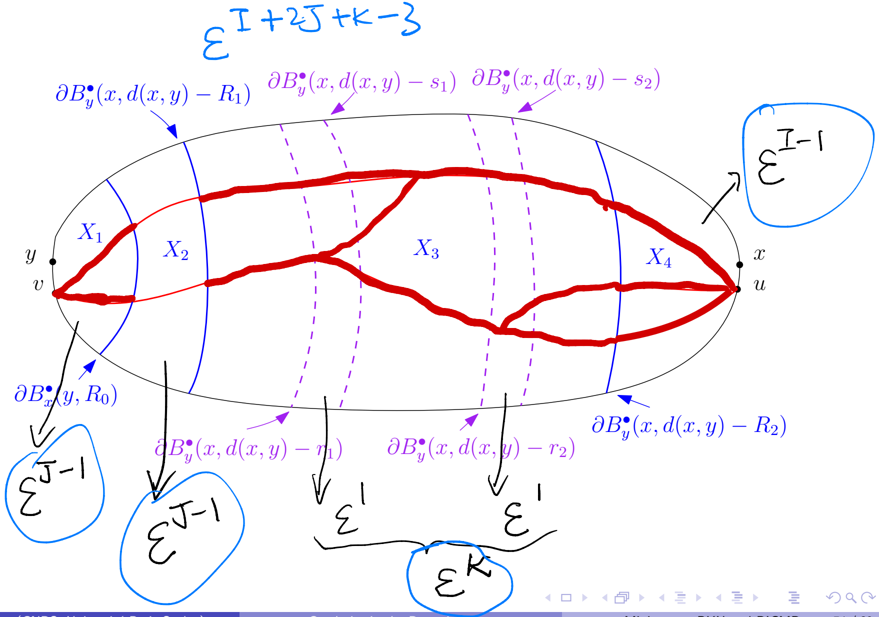
Topology of geodesics between a pair of points

- The asymmetry between I and J in Theorem 6 is due to the asymmetry in the definition of a splitting point.
- In the language of [Angel, Kolesnik and Miermont '17], if u and v are connected by a (j,k)-normal network, then $I=j, J=k$ and $K=j-1$. Theorem 6 implies that the dimension of such pairs (u,v) is at most

$$11-(j+2k+(j-1))=12-2(j+k),$$

equal to the dimension computed in [Angel, Kolesnik and Miermont '17].

Proof of Theorem 6



Number of geodesics between a pair of points

Let Φ_i be the set of pairs of distinct points in $\mathcal{S}$ that are connected by exactly i geodesics.

Theorem 7 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. The set Φ_i is empty if $i \geq 10$. For $1 \leq i \leq 9$, we have

$$\dim_{\mathrm{H}}(\Phi_1) = 8, \quad \dim_{\mathrm{H}}(\Phi_2) = 6, \quad \dim_{\mathrm{H}}(\Phi_3) = 4, \quad \dim_{\mathrm{H}}(\Phi_4) = 4$$
$$\dim_{\mathrm{H}}(\Phi_5) = 2, \dim_{\mathrm{H}}(\Phi_6) = 2, \dim_{\mathrm{H}}(\Phi_7) = 0, \dim_{\mathrm{H}}(\Phi_8) = 0, \dim_{\mathrm{H}}(\Phi_9) = 0.$$

The sets Φ_7, Φ_8, Φ_9 are countably infinite. For all $1 \leq i \leq 9$, the set of points $u \in \mathcal{S}$ such that there exists $v \in \mathcal{S}$ with $(u, v) \in \Phi_i$ is dense in $\mathcal{S}$.

Number of geodesics between a pair of points

The **upper bounds** in Theorem 7 follow from Theorem 6 and the optimal configurations.
The **lower bounds** in Theorem 7 and **the description of** Φ_7, Φ_8, Φ_9 are obtained as follows :

- For $i \in \{2, 3, 4, 6, 9\}$: By [Angel, Kolesnik and Miermont '17], the dimension of the pairs of points connected by a (j, k)-normal network is $12 - 2(j + k)$. Since (j, k)-normal networks $\subseteq \Phi_{jk}$, this gives the lower bounds of $\dim_{\mathrm{H}}(\Phi_i)$ for $i \in \{2, 3, 4, 6\}$.
  It was shown in [Angel, Kolesnik and Miermont '17] that there is a dense and countably infinite set of points connected by a $(3, 3)$-normal network. Theorem 6 shows that there do not exist other configurations leading to 9 geodesics.
- For $i \in \{5, 7, 8\}$, the optimal configurations are not normal networks. We will use different techniques to deal with these cases.

Approximation by geodesics between typical points

Theorem 8 (Miller, Q. '20)

The following holds for μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$. For every geodesic $\eta : [0, T] \to \mathcal{S}$, every $0 < s < t < T$ and $\varepsilon > 0$, there exists $\delta > 0$ such that every geodesic $\xi : [0, S] \to \mathcal{S}$ with $\xi(0) \in B(\eta(s), \delta)$ and $\xi(S) \in B(\eta(t), \delta)$ satisfies

$$\xi([\varepsilon, S - \varepsilon]) \subseteq \eta \quad \textit{et} \quad \eta([s + \varepsilon, t - \varepsilon]) \subseteq \xi.$$

We can choose the points $\xi(0)$ and $\xi(S)$ to be ν-typical, which implies that every geodesic of the Brownian map can be arbitrarily well approximated by a geodesic between typical points.

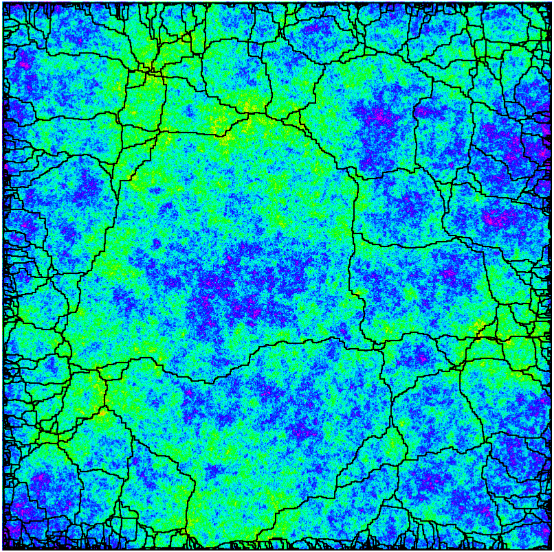
Geodesic frame

The geodesic frame $\mathrm{GF}(\mathcal{S})$ is the union of all the geodesics in $\mathcal{S}$ minus their endpoints.

- Clearly, $\dim_{\mathrm{H}} \mathrm{GF}(\mathcal{S}) \geq 1$.
- Conjecture : $\dim_{\mathrm{H}} \mathrm{GF}(\mathcal{S}) = 1$. [Angel, Kolesnik and Miermont '17]

Corollary 9 (Miller, Q. '20)

For μ_{BM} a.e. instance of Brownian map $(\mathcal{S}, d, \nu)$, we have $\dim_{\mathrm{H}} \mathrm{GF}(\mathcal{S}) = 1$.



- Gaussian free field h. The color represents the height of h.
- The metric of $\sqrt{8/3}$-LQG is given by

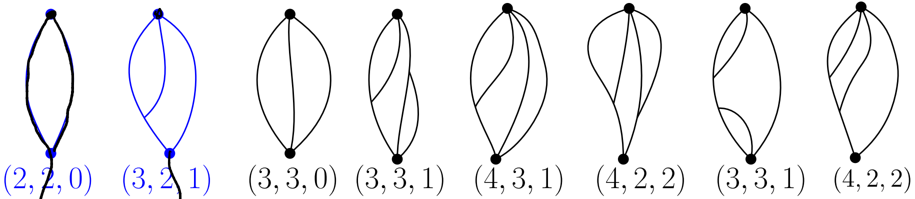
$$e^{\sqrt{8/3}h(x)}(dx^2 + dy^2).$$

- The length of each path P is given by

$$\sum_{x \in P} e^{\sqrt{8/3}h(x)/4}.$$

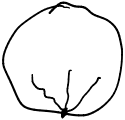
Open questions

- Establish the dimension lower bound for the following configurations (the dimension upper bound is given by $11-(I+2J+K)\geq 0$).



  Do the six last configurations exist ?

- Do there exist 5-star points ?
- Geodesics to the boundary of the Brownian disk.



- Recent works [Gwynne '20] and [Gwynne, Pfeffer, Sheffield '20] prove the analogues of [Le Gall '10] and [Angel, Kolesnik, Miermont '17] for the γ-LQG for $\gamma \in (0,2)$.
  The analogue of our results remain open for the LQG. We believe that a proof can be established following the same strategy, using GFF, but things can get even more technical.

Thank you very much for your attention !