The Brownian loop-soups

MSRI summer school
Random Conformal Geometry

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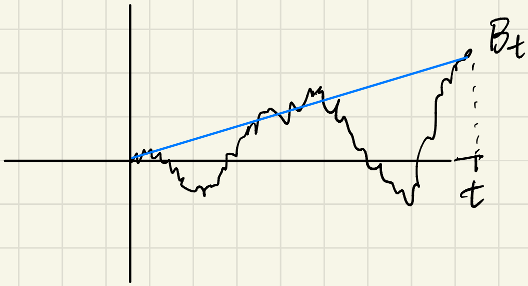
Lecture I. Introduction to the Brownian loop-soups



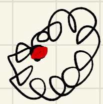
$W_s := B_s - sB_t \quad : s \in [0, t]$

Brownian bridge.

Brownian bridge $a \Leftrightarrow b$

$$\widetilde{W}_s := W_s + a + \frac{(b-a)s}{t}$$

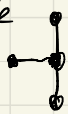
2D: $\quad X_t + iY_t$



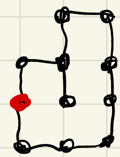
- Brownian loop-soups were introduced by Lawler-Werner [LW04]
- Markovian loops by Symanzik [Sym68]

1. Random walk loop-soup.

Random walk on $\mathbb{Z}^d$



Random walk loop



$$\ell = (z_0, \cdots, z_n) \quad z_i \sim z_{i+1} \quad 0 \le i \le n-1$$
$$z_0 = z_n$$

length $|\ell| = n$

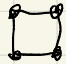
unrooting: $(z_1, \cdots, z_n, z_0) \sim (z_0, \cdots, z_n)$

"Goal": Define a measure μ on unrooted loops such that $\mu(\ell) = (2d)^{-|\ell|}$

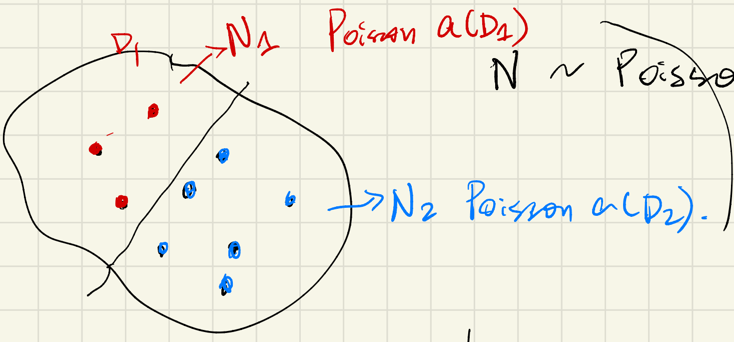
Randomwalk loops were defined by Lawler-Ferreras [LTF07] and were proved to converge to Brownian loops.

- A measure $\tilde{\mu}$ on rooted loops $\tilde{\mu}(\ell) = \frac{(2d)^{-|\ell|}}{|\ell|}$
- $\tilde{\mu}$ induces a measure μ on unrooted loops
  - most often $\mu(\ell) = (2d)^{-|\ell|}$
  - If ℓ is J times the same loop, then

    $\mu(\ell) = \frac{(2d)^{-|\ell|}}{J}$ → non-existing in continuous time.

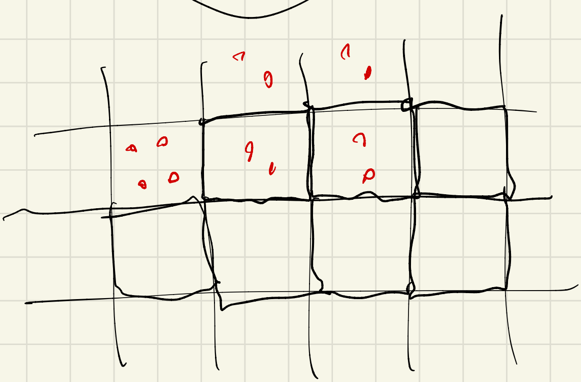
3× 

Random walk loop-soup A collection Γ of loops which is a Poisson Point Process (PPP) with intensity $c\mu$, where $c > 0$ is called the intensity of the loop-soup.



$N \sim$ Poisson $a(D)$

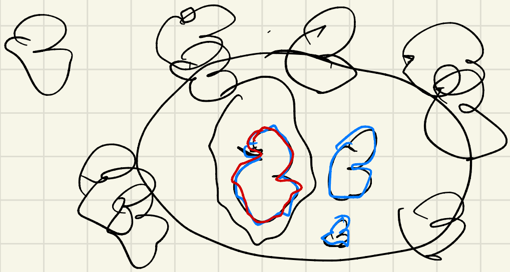
$\oplus \to$ Poisson $a(D)$



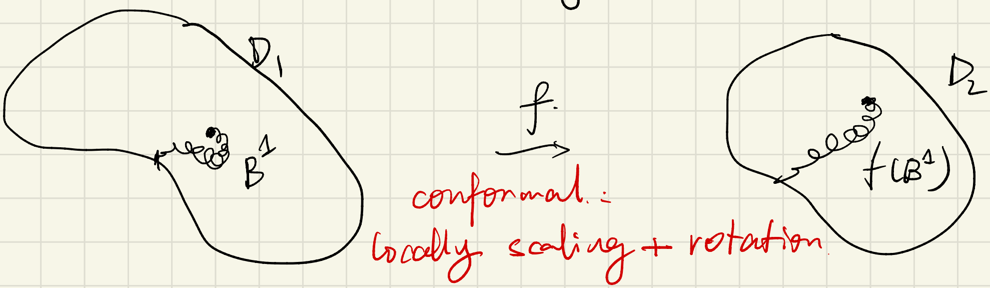
measure σ-finite

For $D \subseteq \mathbb{Z}^d$, let μ_D be μ restricted to the loops that stay entirely in D. Let Γ_D be the collection of loops $\gamma \in \Gamma$ s.t $\gamma \subset D$. Then Γ_D is a PPP with intensity $c\mu_D$.

This is called the restriction property.



In dim = 2. Brownian motion satisfies conformal invariance by Lévy [Lév48]



$f(B^1)$ has the same law as a BM in D_2 stopped upon exiting D_2, modulo time reparametrization.

$$s(t) = \int_0^t |f'(B^1(u))|^2 \, du \quad \text{[by Itô]}.$$

2D Brownian loop-soup by Lawler–Werner is conceived to satisfy.

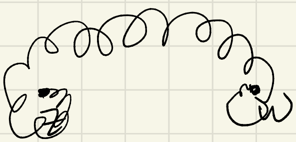
- **Restriction**: $D_1 \subseteq D_2$, Γ_{D_2} restricted to the loops in D_1 is Γ_{D_1}.

- **conformal invariance**: $f: D_1 \to D_2$ conformal
$$f(\Gamma_{D_1}) \overset{law}{=} \Gamma_{D_2}.$$

2. Definition of Brownian loop-soups (2D)

2.1. Brownian excursions

$z, w \in \mathbb{C}, \quad t > 0$



Let $\mu^{\#}(z, w; t)$ be the prob measure of a Brownian bridge from z to w of time length t.

$$\mu(z,w) = \int_0^\infty \underbrace{\frac{1}{2\pi t} \exp\left(-\frac{|z-w|^2}{2t}\right)}_{\text{gaussian density}} \mu^{\#}(z,w,t)\, dt.$$

- infinite mass near $t = \infty$

For $z = w$, we get a measure on rooted loops
$$\mu(z, z) = \int_0^\infty \frac{1}{2\pi t} \mu^{\#}(z, z, t)\, dt$$

- infinite mass near $t = 0, \infty$.

For $D \subseteq \mathbb{C}$, $z, w \in D$, let $\mu_D(z,w)$ be $\mu(z,w)$ restricted to Brownian paths contained in D.

By def $\{\mu_D(z,w)\}$ satisfies restriction.

If $z \neq w$ and ∂D is not harmonically trivial then

$$|\mu_D(z,w)| = 2\, G_D(z,w)$$

where $\Delta G_D(z,\cdot) = \delta_z(\cdot)$ with 0 b.c

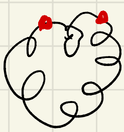
$$G_{\mathbb{H}}(t, w) = \frac{1}{2\pi} \log \frac{|z-\bar{w}|}{|z-w|}$$

$G_D(z,z) = \infty$ so $|\mu_D(z,z)| = \infty$

conformal invariance $f: D \to D'$ conformal.

$$f \circ \mu_D(z,w) = \mu_{f(D)}(f(z), f(w)).$$

2.2. The unrooted Brownian loops.



Brownian measure on unrooted loops:

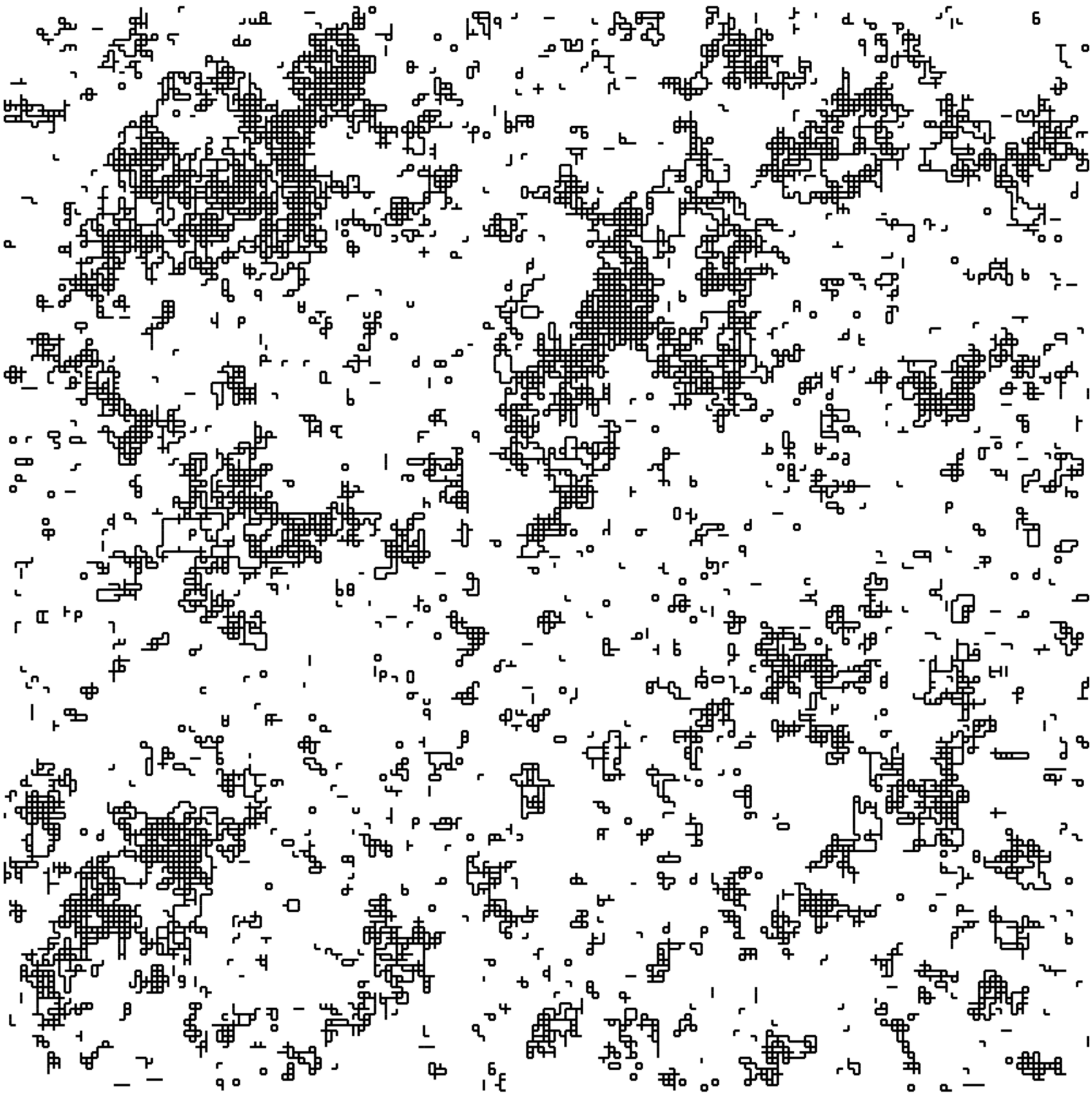
$$\mu^{loop} = \int_{\mathbb{C}} \frac{1}{t_\gamma} \mu(z,z)\,dz = \int_{\mathbb{C}} \int_0^\infty \frac{1}{2\pi t^2} \mu^{\#}(z,z,t)\,dt\,dz.$$

Restriction: obvious from restriction of $\{\mu_D(z,z)\}$.

Conformal invariance: Exercise.

A Brownian loop-soup with intensity $c>0$ is a PPP of intensity $c \cdot \mu^{loop}$.

- infinity many small loops in the neighborhood of every given point $\longrightarrow$ dense.
- locally finite in any bounded domain D. $=$ finitely many big loops in D.

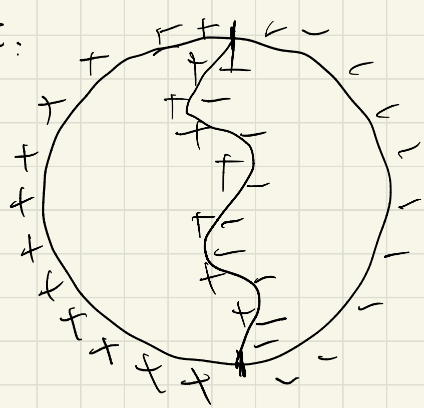


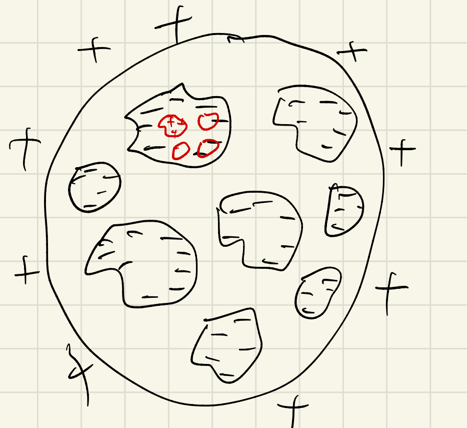
Lecture II. Commuting couplings.

1. Loop-soup and CLE (Conformal loop ensemble)

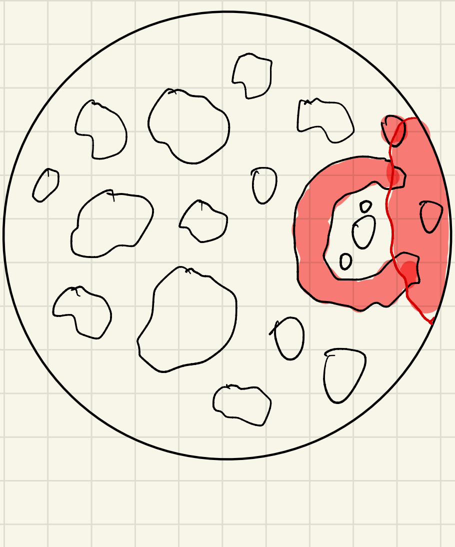
CLE: [She09] by Sheffield: A family of measures depending on $\kappa \in (8/3, 8]$, constucted by SLE_κ
[SW12] by Sheffield and Werner using loop-soups

SLE:





We only look at the regime $\kappa \in (8/3, 4]$ where the loops are a.s. simple and disjoint.



A simple CLE in $\mathbb{U}$ is a random collection Λ of non-nested simple disjoint loops in $\mathbb{U}$.

* locally finite: $\forall \varepsilon > 0$, finitely many loops of diameter $\geq \varepsilon$.
  $\Rightarrow$ countably many loops.
* conformally invariant: law

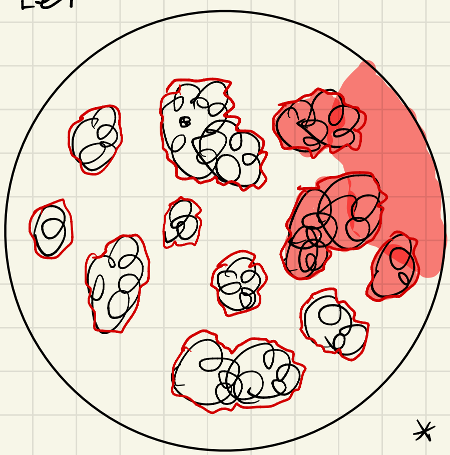
is invariant under all conformal maps $\mathbb{D} \to \mathbb{D}$

$\Rightarrow$ we can define a CLE in any simply connected domain D.

*. spatial Markov property: For any A s.t. $\mathbb{D} \setminus A$ is still simply connected, let A^* be the union of A with all the filled loops in Λ that intersect A. Then in each connected component of $\mathbb{D} \setminus A^*$, we have a loop ensemble which is distributed as Λ, and is indep of A^*.

[SW12]. these 3 axioms characterize (simple) CLE and identified them with the CLEs in [She09]

Let Γ be a Brownian loop-soup with intensity $c > 0$.



We say two loops γ_1, γ_2 are in the same cluster if there is a finite chain of loops $\gamma_1 \sim \eta_1 \sim \cdots \eta_n \sim \gamma_2$

$\sim$: overlap.

[SW12].

* If $c > 1$, there is a.s. a unique cluster in Γ.

*. If $c \in (0, 1]$, then there are a.s infinitely many clusters in Γ. The outer boundaries of the outermost clusters

in Γ are distributed as a CLE_κ, where

$$c = \frac{(3\kappa-8)(6-\kappa)}{2\kappa}$$

critical $c=1$
$\Leftrightarrow \kappa=4$.

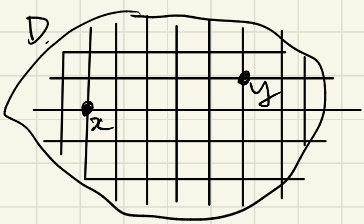
Proof idea: * locally finite: loop-soup is locally finite.

*. conformal invariance is built in.

*. spatial Markov property of loop-soups

*. Percolation: approximated Brownian loops by random dyadic squares $\Rightarrow c_0 \in (0,\infty)$.

2. Loop-soup and Gaussian free field (GFF)



Let D^δ be a discretization of D.

A (discrete) Green's function on D^δ (V,E) is given by

$$G(x,y) = \mathbb{E}_x\Big[\sum_{n=0}^{\tau} \mathbb{1}\{X_n = y\}\Big]$$

X_n is a SRW on D^δ started from x.
τ is the first time X_n exits D^δ.

A GFF h on D^δ with 0 b.c. is a centered Gaussian vector indexed by V with covariance

$$\mathbb{E}[h(x)h(y)] = G(x,y).$$

Lemma. Let w be a real positive vector indexed by V. Then $\mathbb{E}\left[\exp\left(-\sum_{x\in V} w(x)\frac{h(x)^2}{2}\right)\right] = \sqrt{\frac{\det G_w}{\det G}}$.
where $G_w = (W + G^{-1})^{-1}$ and W is the diagonal matrix with entries $W(x,x) = w(x)$ for $x \in V$.

A GFF h in D with 0 b.c. is a distribution in D.

$$\int_D h(x) f(x)\,dx.$$ is a centered Gaussian

doesn't exist.

with variance

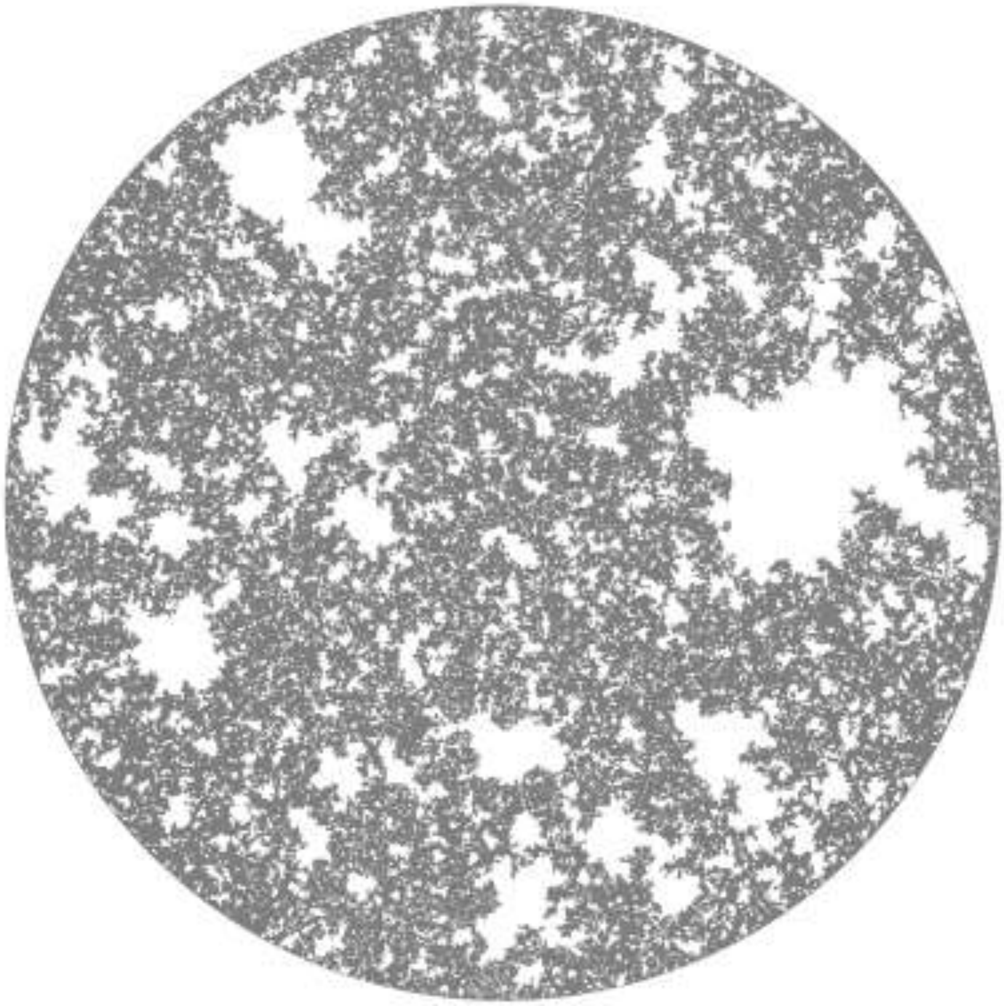
$$\int_D f(x) f(y) G(x,y)\,dx\,dy.$$

The continuous GFF does not have pointwise value

$$\frac{1}{\varepsilon^2}\int_{B(x,\varepsilon)} h(x)\,dx$$ diverges as $\varepsilon \to 0$.

The square of the GFF is a distribution $:h^2:$

$$:h^2:(f) = \lim_{\varepsilon\to 0}\int \frac{1}{\pi\varepsilon^2}\left[h(B(x,\varepsilon))^2 - \mathbb{E}\left(h(B(x,\varepsilon))^2\right)\right] f(x)\,dx.$$

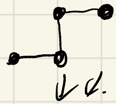


Simulation of CLE_4 by David Wilson

2.2. Occupation time field of the loop-soup

Isomorphism theorem

Discrete



at each vertex, it spends an exp time with parameter 1.

We can define an (unrooted) loop measure μ.

Lemma.

$$\mu\left[1 - \exp\left(-\sum_{x \in V} w(x) L(x)\right)\right] = \log\left(\frac{\det G}{\det G_w}\right).$$

where $L(x)$ is the time that a loop spends at x.

Occupation time field of discrete loop-soup at critical intensity

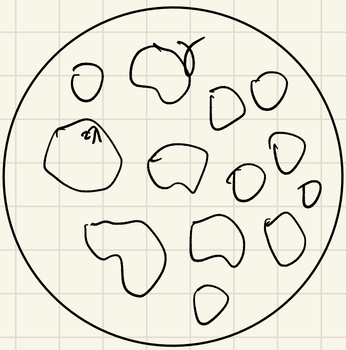
$= h^2/2$.

Isomorphism

Le Jan [LJ11].

3. GFF / CLE_4 coupling by Miller–Sheffield.

Let h be a GFF in $\mathbb{D}$.



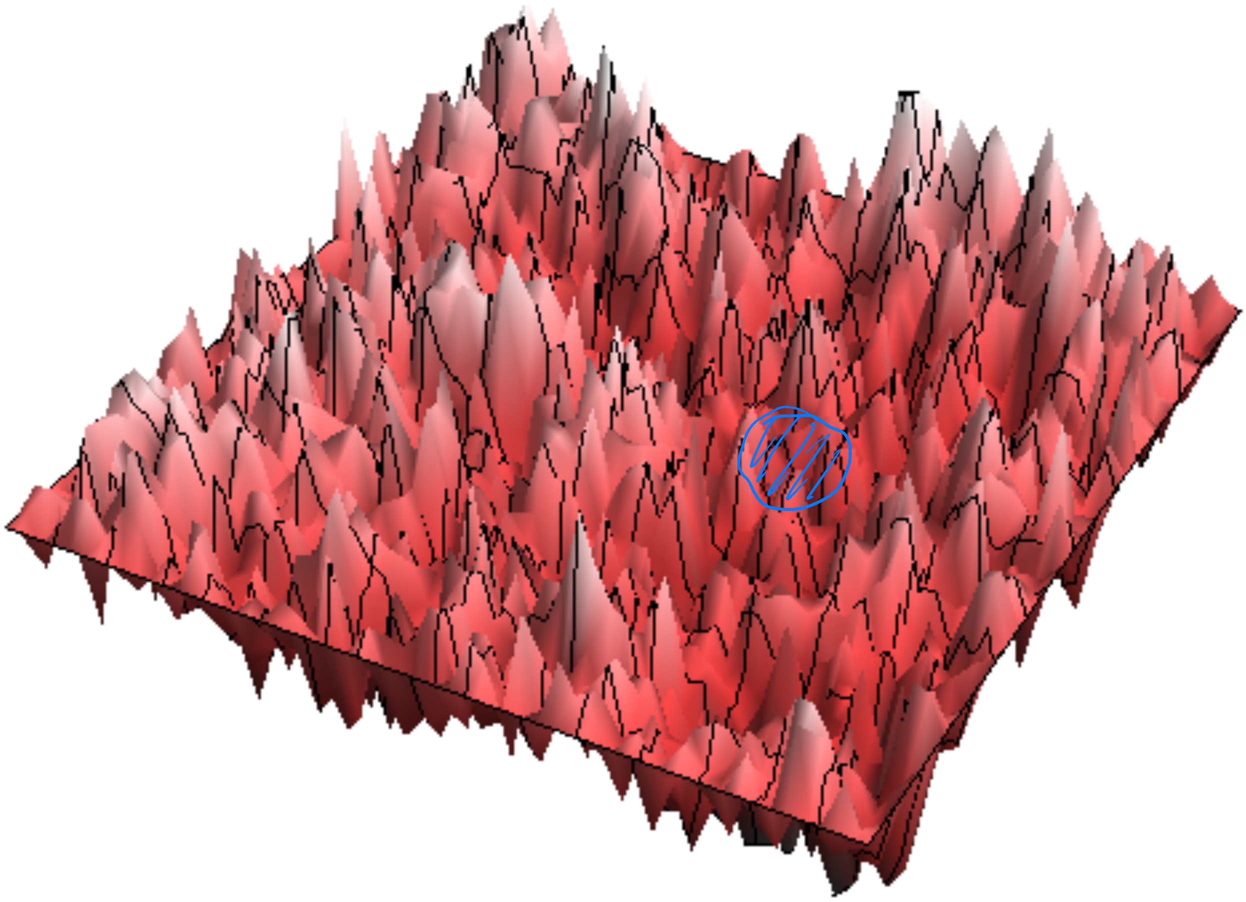
h uniquely determines a set Λ of loops s.t.

* $h|_{O(\gamma)}$ is a GFF with b.c. $\pm 2\lambda$ $\left(\lambda = \sqrt{\pi/8}\right)$
* $h|_{O(\gamma)}$'s are indep for diff γ's.

Λ turns out to be CLE_4.

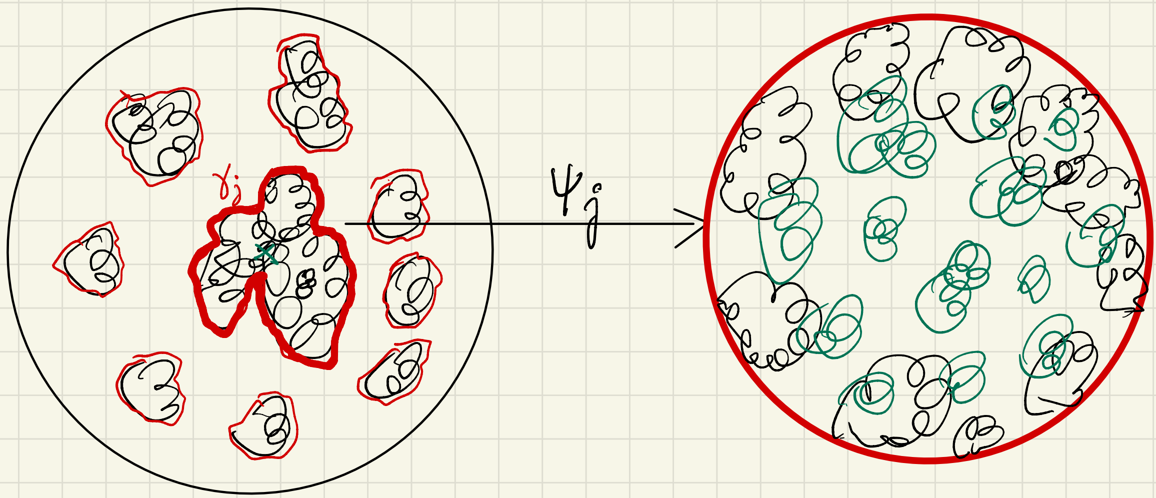
$$
\begin{tikzcd}
\text{critical } (c=1) \text{ loop } \cancel{\text{sop}} \arrow[rr, leftrightarrow, "\text{Sheffield–Werner}"] \arrow[dr, leftrightarrow, "\text{Le Jan}"'] & & CLE_4 \arrow[dl, leftrightarrow, "\text{Miller–Sheffield}"] \\
& \text{GFF} &
\end{tikzcd}
$$

Theorem [QW19].2015 The three couplings commute.



Lecture III. Decomposition of B. loop-soup clusters

Λ B. loop-soup $c \in (0, 1]$



Let $\Gamma = (\gamma_j, j \in J)$ be the collection of outerboundaries of outermost clusters. Let O_j be the domain encircled by γ_j. Let ψ_j be a conformal map from O_j onto $\mathbb{U}$.

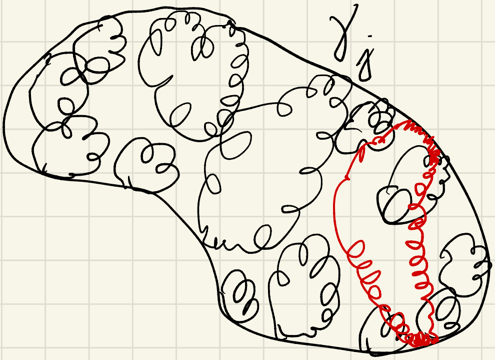
Theorem [QW19] 2015 Conditionally on Γ, $\Lambda \cap \overline{O}_j$ for $j \in J$ are indep of each other. $\psi_j(\Lambda \cap \overline{O}_j)$ is invariant under all conformal maps $\mathbb{U} \to \mathbb{U}$ and is indep of Γ.

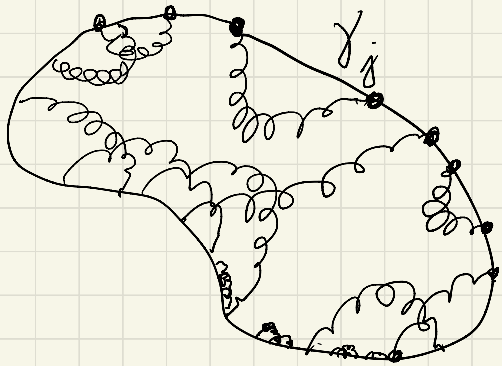
$\Lambda \cap \overline{O}_j$ can be decomposed into two (conditionally) indep parts: ① a collection of loops in $\overline{O}_j$ that touch γ_j

② a Brownian loop-soup with intensity c in O_j

Proof: spatial Markov property and conformal invariance of the loop-soup.

If $c=1$. The trace of ① is distributed as the trace of a PPP of Brownian excursions with intensity $1/4$ in O_j away from the boundary.



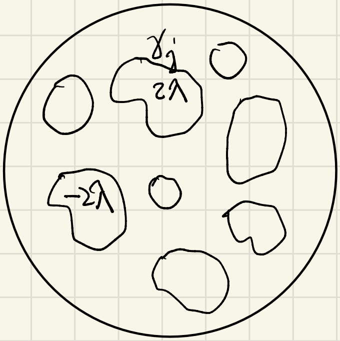


* each loop bounces on γ_j infinitely many times
  $\Longleftrightarrow$ infinitely many small excursions.

*. PPP $\rightarrow$ "indep" excursions. specific to $c=1$.

Proof idea: coupling loop-soup — CLE_4

$$\text{occup. time } \mathcal{T} \swarrow \text{loop-soup} \searrow \text{GFF} \nearrow CLE_4$$



$h|_{O_j}$ is a GFF with b.c. $\underbrace{\epsilon(\gamma_j)}_{\in\{-1,1\}} 2\lambda$

$$T|_{O_j} = \text{``}(h|_{O_j})^2/2\text{''}$$

$\neq :(h+u)^2:$

<u>Lemma</u> (Sznitman [Szn13]) In D, h is a 0 b.c. GFF

$$\frac{:h^2: + 2uh + u^2}{2} = \frac{1}{2}:h^2: + \mathcal{T}_{ku^2}$$

where $\mathcal{T}_{ku^2}$ is the occup time of a PPP of Brownian excursions with intensity ku^2.

$$T|_{O_j} = \underbrace{T_{\text{loop-soup in } O_j}}_{2} + \underbrace{T_{\text{PPP of B.E in } O_j}}_{1}$$

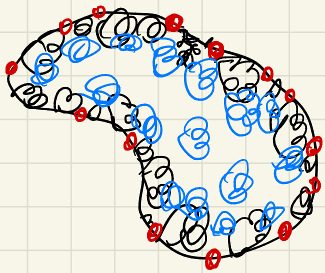
$\Rightarrow$ occup time field of the collection of loops in O_j that touch the boundary $=$ occup time of PPP of B.E in O_j

$\Rightarrow$ their trace have the same law.

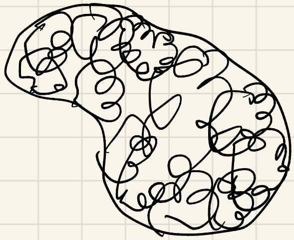
[QW18] The law of a point process of B.E is determined by the law of its trace.

<u>Q1.</u> What about $c \in (0, 1)$?

[Qia 19] For all $c \in (0, 1]$, the boundary-touching loops satisfy a certain restriction property with parameter α.



$c \in [14/15, 1]$

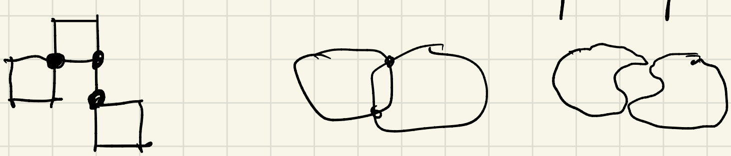


$c \in (0, 14/15]$

The greater c is, "less" points are on the boundary of a cluster.

Q2. For $c = 1$, given the excursions, how to hook them back into loops? Any randomness involved?

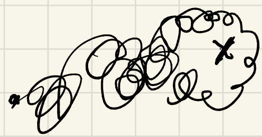
[Wer 16]. Resampling property in the discrete for the critical loop-soup.



Q3. Are there double points on the boundaries of clusters in the B. loop-soup?

Multiple points in general.

* a given $z \in D$ is a.s. not visited by any loop.



*. The union of loops in Γ is dense and has dim 2.

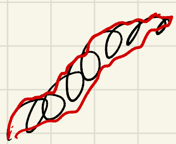
* The set of n-tuple points have dim 2 [Tay66].

Multiple points on cluster boundaries.

*. The H. dim of cluster boundaries

$$1 + \kappa/8 > 4/3$$

* $4/3$ is the dim of the Brownian frontier.



Mandelbrot conjecture

proved by [LSW01a] [LSW01b].

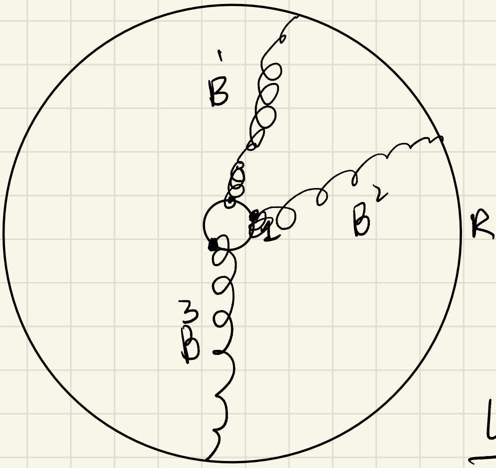
* Most points on the bdy of a cluster do not belong to any loop.

*. There exist points on a cluster bdy that belong to at least one loop.

*. No easy answer for Q3.

For the Brownian motion, the dimension of double points on the frontier is $\frac{\sqrt{97}+1}{24}$ [KM10].

computed using Brownian disconnection exponent.



Let p_n^R be the prob that $B^1[0,T_R^1] \cup \cdots \cup B^n[0,T_R^n]$ does not disconnect 0 from ∞.

Then $p_n^{RS} \leq p_n^R \cdot p_n^S$ $\quad R, S > 1$.

$$\frac{\log p_n^R}{\log R} \xrightarrow[R\to\infty]{} -\eta(n).$$

conjectured by [DK88]

computed by LSW using SLE

$2-\eta(2)$

$2-\eta(4)$

Generalized disconnection exponent [Qia21]

$$\eta_c(\beta) = \frac{1}{48}\left[\left(\sqrt{24\beta+1-c} - \sqrt{1-c}\right)^2 - 4(1-c)\right]$$

using radial hypergeometric SLEs.

[Qia 21, Section 1.4]. clarification

[GLQ] 21+

$2 - \eta_c(2)$ $\quad$ $2 - \eta_c(4)$.

decreasing in c.

$c = 1$. dim double points $= 0$.